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A REVIEW OF CURRENT IDEAS ON THE TEACHING OF MATHEMATICS[1]

The subject of elementary mathematics has been the last to respond to improved pedagogical methods. Recently, however, there has been a general awakening of interest in methods of mathematical instruction which seems to be universal in its scope, and to have originated simultaneously in France, Germany, England and the United States. Although the discussions have been widely varied, there is practical unanimity on the point that mathematical instruction should be less formal and more practical, and should constitute simply an extension of the ordinary experience of the child.

Although the ideas recently expressed are almost wholly deductions from experience, they seem, for the most part, to have a scientific basis. They recognize, namely, that the child learns as the race has learned, by proceeding from concrete and familiar facts to the general laws underlying all

[1] A summary of reports on special topics, assigned in connection with a course for teachers on the history and teaching of mathematics, given by Professor S. E. Slocum at the University of Cincinnati, 1907–8. The class, part of whose work is here presented, was composed as follows: Charles Otterman, A.B., professor of mathematics, Woodward High School, Cincinnati, Ohio; Albert Schwartz, A.B., principal Linwood School and principal East Night High School, Cincinnati, Ohio; Benjamin H. Siehl, A.B., teacher eighth grade, Morgan School, Cincinnati, Ohio; Charles H. Siehl, A.B., teacher eighth grade, Garfield School, Cincinnati, Ohio; Jesse K. Dunn, principal Highlands School, Cincinnati, Ohio; F. L. Williams, A.M., principal William Grant High School, Covington, Kentucky; Howard Hollenbach, B.Sc., instructor in science, Lockland High School, Lockland, Ohio.

experience. The historical parallel, in fact, is strikingly manifest in the form, content and sequence of development of elementary mathematics as proposed in these reforms.

GERMAN REFORMS[2]

The nineteenth century may be divided into three periods as regards the form and content of mathematical instruction in Germany. In the first period, extending from 1800 to 1870, mathematical instruction was a mixture of the pure and applied. Ideals were high, efforts were directed toward awakening individual ability, and attempts were made to teach more than is required at present. The candidate for the position of teacher of mathematics must be a specialist capable of original investigation, and as a result we find such names as Grassmann, Kummer, Plucker, Weierstrass and Schellbach.

The opening of the second period, 1870–1890, was signalized by the victory over France, and the assumption by Germany of a more important international position. This period was marked by a separation of pure and applied mathematics. In the schools the feeling prevailed that the development of the especially gifted pupil was not to be sought so much as that of the average pupil, and consequently greater interest was manifested in the method of instruction. Instead of the early system, a desire was expressed for a systematic graded course in mathematics, keeping in view the ability of the constantly developing pupil. Drawings and models were demanded, problems were so stated and aids so given that the pupil might see space relations and not depend so largely on the logic of the ancient Greeks. This was a direct result of the teachings of Pestalozzi and Herbart. In this period the standard for teachers was lowered, and there was only required a knowledge sufficient to solve problems of moderate difficulty.

The third period, beginning with 1890, seems to be characterized by a tendency to again associate pure and applied mathematics. That is to say, a teacher is required to be thoroughly familiar with pure mathematics and at the same time to have an extensive knowledge of its applications.

For many decades the value of mathematical training was thought to lie in its formal discipline. Before the revival of learning it was the utilitarian factor which was emphasized, while in the last few decades the majority have reached a more comprehensive notion. Recently Professor Felix Klein, the greatest of living German mathematicians, has shown a deep interest in the problems of the schools, and has taken an active part in their discussion. His views are typical of modern German scholarship, and form the basis of the reforms instituted and proposed.[3]

According to Professor Klein, mathematical thought should be cherished in the schools in its fullest independence, its content being regulated in a measure by the other problems of the school; in other words, its content should be such as to establish the liveliest possible connection with the various parts of the general culture which is typical of the school in question. Here, then, it is not a question of the method of teaching, but rather of the selection of material from the great mass furnished by elementary mathematics.

Much of the material of instruction, although interesting in itself, lacks connection and is wholly or partially isolated, thus affording only a faulty and indirect

[2] For a more detailed account, see article by Charles Otterman entitled "A Review of Klein's Attitude on the Teaching of Mathematics," SCIENCE, September 13, 1908.

[3] "Neue Beiträge zur Frage des Mathematischen und Physikalischen Unterrichts an den Hoheren Schulen," Klein und Riecke, Teubner, 1904.

preparation for a clear understanding of the mathematical element of modern culture. This element clearly rests on the idea of function, both geometrical and analytical, and this idea should be made the center of mathematical instruction. In fact, Klein's chief thesis is that beginning with the Untersecunda and continuing in regular methodical steps, the geometrical conception of a function should permeate all mathematical instruction. In this is included the consideration of analytic geometry and the elements of differential and integral calculus.

The ground to be covered depends largely upon the ideal of the school. While the formal side must not be overlooked, and a thorough knowledge of the processes must be obtained, the principal aim should be to give a clear conception of the fundamental ideas and their meaning.

Mathematical instruction on the level at which it is carried on in the upper classes of the higher schools has existed in Germany since about the beginning of the eighteenth century. In this early formative period calculus was not considered elementary, for it was the possession of only a few investigators of the highest type. Cauchy's great work on differential and integral calculus appeared in 1821, but the schools had already been led into certain channels, and it was not possible to divert them toward a subject which was only in the process of formation. In fact, calculus was considered as a sort of witchcraft, and has ever since been regarded with suspicion.

The official course of study of 1900, however, showed a tendency in the opposite direction, so that Klein believes that advantage should be taken of this favorable attitude to place that which has taken centuries for preparation upon a generally recognized basis.

As a matter of fact, the ideas underlying the calculus are actually taught in many schools. In a few Ober Real schools they are regularly taught as calculus, but in the majority of schools they are taught in the most round-about manner. In fact, students are actually taught to differentiate and integrate as soon as occasion for the same arises, but the terms differential and integral are avoided.

Klein's opinion is that instead of making instruction in calculus in those grades whose work demands its employment incidental, desultory and generally unsatisfactory, it should be made the central idea of instruction, and the other ideas and work grouped around it. At present, calculus is made the beginning of higher mathematics and is accompanied by a revolution in thinking. Klein's suggestions aim to obviate this difficulty by gradually accustoming the pupil to those methods of thinking which later predominate.

THE PRUSSIAN SYSTEM

The Prussian schools are probably the most efficient, in point of actual instruction, of the entire German school system, and for this reason deserve special consideration.

Although but 1.2 years of the nine school years are given to mathematics as compared with 2.1 years in this country, the Prussians accomplish fully as much as, if not more than, our American schools, with a saving of .9 of a year, or .1 of the total time of instruction for the nine years.

The three main causes of the excellence of the Prussian work in mathematics are the central legislation and supervision, the thorough preparation of the teachers and the systematic methods of instruction. Each of these is a result of the preceding; for well-prepared teachers are likely to use good methods of instruction and the thorough preparation of teachers is sure to be

required when the legislative and executive authority is vested in experienced educators of the highest order. It might be sufficient to say that the strength of the Prussian system is due to the fact that for two thirds of a century the work has been centered in a single source of authority for the entire kingdom.

In American schools considerable loss is due to the fact that during the nine years which correspond to the German Gymnasium, every pupil must make two changes, viz., from the grades to the high school and thence to college. The work of these nine years is thus carried on under different circumstances, with different names and methods, and under teachers differing in their preparation; and instruction necessarily suffers in consequence.

The Germans appreciate the fact that the teacher can do his best only in an atmosphere of financial and mental tranquility, and while insisting on high standards and severe tests at the outset, assure a tranquil career to those who have given evidence of their fitness, by a regular system of promotions and pensions.

In the common schools, the aim is to train good and faithful citizens, the process being called "Erziehung" (bringing up).

In the higher schools, the aim is to impart learning, and to turn out educated and cultured men. The process is called "Unterricht" (instruction), and leads to privileges and responsibilities before the civil and military law. Attendance upon the common schools is regarded as the duty of those not having better opportunities and is enforced by the state, whereas attendance upon a higher school is considered a privilege and the state may restrict the number of persons admitted.

The work in mathematics done in the higher schools covers practically the same ground as our course to the close of the freshman year in college. The general aim of the instruction is facility of calculation with numerical quantities and their application to the usual circumstances of everyday life, algebra to quadratics, plane and solid geometry, plane trigonometry, the idea of coordinates and some of the fundamental properties of conic sections. In all of these subjects an effort is made to impart an intelligent knowledge of the theorems, as well as skill and facility in their application.

The entire mathematical education of the boy from the elements of arithmetic to those of analytic geometry takes place in one institution under one management guided by the close supervision of the same director and under the tuition of men of the same scientific training, who are colleagues, working in close contact, with opportunities for intimate interchange of ideas.

The aim of the teaching of arithmetic is to secure facility in operations with numbers. In order that it may be in harmony with the following algebraic instruction, and prepare for it, the reviews of the fundamental operations and the treatment of fractions is based upon mathematical form, and the handling of parentheses is continually practised. In fractions the pupil is taught to operate with fractional parts as concrete things, and facility in computation is maintained by continued numerical exercises in the algebraic instruction in the following classes.[4]

In the Real Gymnasium the scope of instruction in algebra includes the proof of the binomial theorem, and the solution of equations of the third degree; plane geometry, including the theory of harmonic points and pencils and axes of symmetry;

[4] For a description of courses see "The Teaching of Mathematics in the Higher Schools of Prussia," J. W. A. Young, Longmans, 1900.

solid geometry, and the elements of descriptive geometry; plane and spherical trigonometry; introduction to the theory of maxima and minima; plane analytic geometry.

In addition to this the Oberrealschule requires the most important parts of algebraic analysis. Equations of the fourth degree, and the approximate numerical solution of algebraic and transcendental equations may be taken up at the option of the instructor. In all these topics the aim is to give practise in the application of the theorems, as well as to lead to a mastery of the proofs. Considerable stress is also laid upon the oral solution of easy exercises.

The German higher school system speaks with no uncertain tone of the practical advantage of beginning both algebra and geometry early; and continuing their study simultaneously through a number of years.

The method of instruction in Prussia may be styled the genetic method in that it does not require any previous preparation of the pupil for the work of the hour. The quantity of home work is kept as small as possible, and material that has not been thoroughly explained in the class is never assigned to be studied privately by the pupil. A text-book is officially required but is seldom used, and at most the pupil is only referred to it after class treatment, and not before. The genetic method has been but little used in America, but there is a strong trend toward the use of the heuristic method, which is somewhat similar. It resembles the genetic method in its marked effort to keep the pupil thinking for himself, but differs from it in that the class is the working unit in the genetic method, whereas the pupil is the unit in the heuristic method. This difference may be said to be characteristic of the German and American methods of mathematical instruction. In Germany the class works as a whole under the guidance of an instructor. In America the individual pupil demonstrates, explains and asks and answers questions, while the others listen to him.

While the results of the genetic system as applied to the Prussian schools amply justify its use with German pupils, it by no means follows that the method is of universal application, and could be used to advantage in this or other countries. The other features of the German school system, however, relating to the high standard of technical excellence in the teaching profession, and the correlation and continuity of instruction, are of the deepest and most general pedagogical significance.

ENGLISH REFORMS: THE PERRY MOVEMENT

The English reforms are largely based on the so-called "Perry movement," which originated from a paper on the teaching of mathematics read by Professor John Perry at a meeting of the British Association in Glasgow in 1901.[5] In this paper Professor Perry maintained that *usefulness* should be the criterion for determining what subjects should be taught to children, and in what way they should be taught.

The present methods of teaching mathematics keep students so long upon the knowledge already organized that there is neither time nor enthusiasm for undiscovered fields. Absolute correctness and philosophical insight receive too much attention in the teaching of elementary subjects. For example: the first four books of Euclid might be assumed, or accepted partly by faith and partly by trial, and the sixth book regarded as axiomatic. In this way the student might begin his work earlier and much more be accomplished. What is

[5] "Discussion on the Teaching of Mathematics," edited by Professor Perry. British Association Meeting, Glasgow, 1901. Macmillan, 1902.

omitted would be compensated for by an increase in youthful enthusiasm and the development of individuality and inventiveness.

It is well recognized that the study of natural science is essential to all courses of study. Such study, however, is impossible without a practical, working knowledge of mathematics, and facility in its application to problems in engineering and general science. The elementary study of nature requires skill in computing with logarithms; knowledge of, and power to manipulate, algebraic formulas; the use of squared paper, and the methods of the calculus. Professor Perry believes that boys may not only become skilful in the use of these instruments, but will learn them with pleasure. He also asserts that the men who are teaching orthodox mathematics are not only destroying what power to think exists, but are also producing a dislike and hatred for all kinds of computations, and, therefore, for all scientific studies of nature.

As the basis of his belief that instruction in elementary mathematics should be made more practical, Professor Perry states that

> In the whole history of the world there was never a race with less liking for abstract reasoning than the Anglo-Saxon. Every other race has perfected abstract schemes of government. Here common-sense and compromise are believed in; logical deductions from philosophical principles are looked upon with suspicion not only by legislators but by all our most learned professional men.

All of this indicates that philosophy is certainly not intended for children. This was also the view of the ancient Greeks, who held that only a few men were capable of philosophic insight. Reading, writing and ciphering were at one time regarded as learned studies. However, when they became essential to the correct doing of one's daily work, they were taught quite readily to children without unnecessary philosophy. So the child should learn mathematics without unnecessary philosophy. Omitting this philosophic insight, the average boy may learn much useful mathematics which will serve him all through his life.

Professor Perry's plan lays emphasis upon the following propositions: (1) Experimental methods in mensuration and geometry ought to precede demonstrative geometry, although even in the earliest stages some demonstrative reasoning may be introduced. (2) The experimental methods adopted may be left largely to the teacher. (3) Decimals ought to be used in arithmetic from the beginning. (4) The numerical solution of complex mathematical expressions may be taken up almost as a part of arithmetic, or the beginning of algebra, as it is useful in familiarizing pupils with the meaning of mathematical symbols. (5) Logarithms should immediately follow the theory of exponents. (6) The study of the calculus may precede advanced algebra, advanced trigonometry, or analytical geometry, and may be illustrated by any quantitative study in which the pupil may be engaged.

The course in elementary mathematics suggested by Professor Perry includes the following:

In arithmetic, the use of decimals from the outset; contracted and approximate methods, and rough checks on numerical calculation; the meaning and use of logarithms, including the construction and use of the slide rule; calculation of numerical value from algebraic formulas; extraction of square roots; simplification of fractions; calculation of percentage, interest, discount, etc.

In algebra, the translation of verbal statements into algebraic language; numerical application of formulas; rule of in-

dices; factoring, and practise in general algebraic manipulation.

In mensuration, experimental testing of the rules for lengths of curves, for example, in the case of a circle, by rolling a disk, or by wrapping a string around a cylinder; verification of propositions in Euclid relating to areas by the use of squared paper, by means of a planimeter, by using Simpson's or other approximate rules, or by cutting the area out of cardboard and comparing its weight with that of a piece of known area;[6] rules for volumes of solids verified by their displacement of a liquid.

The experimental work in this subject ought to be taken up in connection with practise in weighing and measuring generally, finding specific gravities, illustrating the principle of Archimedes, determining the displacement of floating bodies, and other elementary scientific work. Good judgment will indicate the relative amount of experimental, didactic and numerical work.

The use of squared paper is especially emphasized. Some of its applications which are mentioned, are the plotting of statistics of general or special interest; study of the curves or lines so obtained, such as the determination of their maximum and minimum points, their rates of increase or decrease, etc.; interpolation, or the finding of intermediate values; probable errors of observation and the correction of same; determination of areas and volumes, as mentioned above; plotting of functions and the graphical solution of equations; determination of the laws between observed quantities.

In geometry, the experimental illustration of important propositions in Euclid, frequently supplemented by demonstration; measurement of angles by means of a protractor; definitions of trigonometric functions and the use of trigonometric tables; solution of right-angled triangles graphically and by calculation; construction of triangles and the experimental determination of their areas; method of locating a point in a plane and in space; the elements of descriptive geometry and vector analysis.

The advanced course proposed consists chiefly in an extension and elaboration of the elementary course. It includes demonstrative geometry, and rules in arithmetic and mensuration stated as algebraic formulas. In trigonometry, the study of special limits such as sin x/x and the derivation of the fundamental formulas and relations of trigonometry. In mensuration, the method of determining the constants in such physical formulas as $pv^m = c$. The course then proceeds to differential and integral calculus and their practical applications; differential equations illustrated by practical problems from mechanics and physics, descriptive geometry and vector analysis.

The reforms proposed by Professor Perry were widely discussed, and were, in general, favorably received. It was not to be expected, however, that the traditional teaching of Euclid in Great Britain would undergo any immediate or radical change, or, in fact, that any innovation of the kind proposed meet at the outset with a cordial reception.

In America, however, Professor Perry's views found ready acceptance and were carried more or less completely into effect. The result was what is now called the laboratory method of instruction which has been developed independently at several places, although along most radical lines at the University High School, Chicago. An outline of the laboratory system, together with a number of typical opinions

[6] This is the way in which Galileo is said to have determined the area of the cycloid.

as to its value and practicability, are given in what follows.

THE LABORATORY METHOD

The works of Klein and Perry mark the beginning of a movement to improve on present methods and make a more direct and pleasant path for the average student in the field of mathematics. The essence of the laboratory method consists in the performing of the bulk, if not all, of the work in the mathematical class room, which should be equipped with laboratory appliances for the graphic, the experimental and the concrete phases of the work. The teacher acts as the director of the laboratory, the pupils work individually or in small groups, and analogies with the work in the physical laboratory are emphasized.

The word *laboratory* undoubtedly came to be used largely from the suggestions received from analogous work in the physics laboratory. In 1886 Safford in his "Mathematical Teaching" said that a mathematical laboratory, although not often mentioned, was a necessity, and should contain such things as relate to ordinary, not purely scientific, measures.[7] Young calls attention to the fact that in the physics laboratory students work singly or in small groups under the general supervision of the instructor, but with direct contact with him for only a few minutes, and that this is a *limitation* of the physical laboratory, and not an advantage.[8]

The advantage of the class recitation over individual or private instruction has been pointed out by W. T. Harris, U. S. Commissioner of Education.[9] The class is the most potent of all instruments in the teacher's hand. He so manages the recitation or class exercise that each pupil learns to see the lesson through the minds of all his fellows, and likewise learns to criticize the imperfect statements made by them through the more adequate comprehension of the teacher. But because in mathematics the instruments are so simple, the benefit of laboratory work may be obtained under *class* direction, thus getting the good features of the laboratory system while avoiding its defects.

The procedure which physicists find best pedagogically suggests a plan for mathematics; namely, not that the mathematical class exercise be supplanted by a mathematical laboratory exercise, but that it be supplemented thereby. The mathematical class exercise should be conducted by some good method as at present, with the usual time allotment. This should then be supplemented by work in a well-equipped mathematical laboratory, either under the direction of the teacher or one or more assistants or both. The pupils should do substantially the same work in the laboratory, and the class exercise should prepare directly for it.

The feeling that mathematics must be made more concrete and must come into closer touch with the realities about the pupil, is growing in Germany, France, England and America; and the influence of the work of Perry can be distinctly marked in the current thought on the European continent.[10]

Professor Moore has defined pure mathematics as a language for the convenient expression and investigation of the most diverse relations of life and nature. The principles of the language are not arbitrary, but are imposed by the phenomena demanding expression.[11] From this it fol-

[7] "Mathematical Teaching," Safford, 1886, Heath & Co., 1896.

[8] "The Teaching of Mathematics," J. W. A. Young, Longmans, 1907.

[9] W. T. Harris, 1897.

[10] "Teaching of Mathematics," Young.

[11] "Cross Section Paper as a Mathematical Instrument," Moore, *School Science and Mathematics*, June, 1906.

lows that mathematics should not be presented ready made. The individual should make his own as the race has done; but not as if the race had never done it. That which is distinctly utilitarian in the course must be thoroughly practical and in accord with modern usage.

This idea has been amplified by Professor W. E. Story, who has pointed out that the education of the individual differs from the life history of the race in that the pupil is made to pass through the essential stages of development without wasting his time on what the experience of former generations has shown to be unessential. In other words, education is selective history, and whatever mode of selection most thoroughly excludes the unessential is most economical, enables the pupil to master the largest amount of what is essential, and gives him most time to devote to exploration of new fields when he has explored the old; that is, leads most rapidly to independent thought, which is the true goal of education.[12]

In America the term *laboratory method* has been coined to name the teaching of elementary mathematics as it would be if remodeled in accordance with the aims and ideals of the Perry movement. The dominating thought is a fuller consideration of the child mind; a sacrifice of the logical to the psychological, or rather a recognition that no method of instruction is truly logical which is not psychological. The keynote is interest, viz., that mathematics must be presented to the child in the most interesting way.[13]

One of the most significant features of this movement is its insistent demand for a closer correlation of subjects; or, more specifically, that mathematics and physics be organized into one coherent whole and no distinction recognized between mathematics and its principal applications. This, as shown in what precedes, is also the trend of ideas in Germany. It is essential that the pupil should be familiar, by way of experiment, illustration, measurement and every other possible means, with the ideas to which he applies his logic; and, moreover, should be thoroughly *interested* in the subject studied.[14]

Following out this idea, the secondary schools should advance independently of the primary ones, and algebra, geometry and physics, including astronomy and mathematical and physical geography, be organized into a four years' course comparable in strength and closeness of structure with the four years' course in Latin. The physics should be practical, and selected by an engineer, and the teacher should be trained in mathematics, physics and engineering. To carry out such a reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics and physics, its principal aim being to develop the spirit of research and an appreciation of scientific methods.

One of the most important suggestions of the English movement is that by emphasizing steadily the practical sides of mathematics, that is, arithmetical computations, mechanical drawing and graphical methods generally, in continuous relations with problems in physics, chemistry and engineering, it would be possible to give very young students a great body of the essential notions of trigonometry, analytic geometry and the calculus. It is especially important that teachers in the primary schools should make wiser use of the foundations laid by the kindergarten. Cross-section

[12] "Unification of Mathematics in the School Curriculum," W. E. Story, *School Review*, 1903, pp. 832–55.

[13] "Mathematics in Commercial Work," E. L. Thurston, *School Review*, 1903, pp. 585–92.

[14] "Discussion on the Teaching of Mathematics," Perry, Macmillan, 1902.

paper, tables and blackboards should be used all through the grades. Drawing and paper folding should lead to intuitional geometry and mechanical drawing, and geometry be closely connected with numerical and literal arithmetic.[15]

As phenomena are observed by the individual, they should be described graphically and also in terms of number and measure. The graphical depiction serves to illuminate the quantitative determination, and *vice versa.*

The following has been suggested as a fairly complete equipment for a mathematical laboratory:[16]

1. Set of drawing instruments, board, T square, triangles (for each pupil).
2. India inks, paper, note-books, cross-section paper.
3. A large, well-lighted room, good drawing desks.
4. Carpenter's tapes, surveyor's tapes, architect's scales.
5. Three-, five-, seven-place logarithmic tables; pupils to choose which to use from accuracy of data.
6. Logarithmic slide rules and computing machine.
7. Surveyor's compass, transit, level, rod, poles.
8. Surveyor's plane table and sextant.
9. Steelyards, balances, pendulums, barometers, thermometers.
10. Force appliances such as pulleys and simple machines.
11. One hundred good texts on arithmetic, algebra, geometry, trigonometry, physics, elementary mechanics and astronomy, including Crelle's multiplication table.
12. A dozen treatises on these subjects, and a few good histories of mathematics and the mathematical sciences.
13. Spherical blackboards, concave and convex.
14. Three plane blackboards for projective and descriptive work in geometry.
15. Mathematical models.
16. Samples of actual engineering and architectural drawings of machines and structures.
17. Gyroscopic tops.
18. A set of Hanstein's models for projective work.
19. Stereopticon and slides.

The laboratory method has been given a thorough trial at the University School, Chicago, and methods have been developed and text-books prepared from the laboratory point of view. The aim or ideal of the work for the first year has been formulated by Professor Myers as follows:[17] (1) to generalize and extend arithmetical notions; (2) to follow up the notions of mensuration into their geometrical consequences; (3) to reconnoiter a broadly interesting and useful field of algebra; (4) to treat, with sufficient completeness for high schools, a large part of what is most practical and useful in elementary algebra. This means postponing the scientific and purely logical aspects of algebra to a later period.

Problems are drawn from arithmetic, mensuration, geometry, physics and elementary mechanics; and the equation is made the starting point and agency for developing the topics considered. The text-book which represents this first year's work is essentially an extensive and varied body of mathematical ideas correlated around an algebraic core.[18] The treatment begins with the informal methods of inductive arithmetic, passes to the uses of the equation and its transformations, and by degrees assumes a deductive character. Practical problems of a constructional or mensurational character have been found to appeal to first-year pupils with greater drawing force than any other problems of the text.

[15] "On the Foundations of Mathematics," Moore, *School Review*, 1903, pp. 521–38.

[16] "The Laboratory Method," C. W. Myers, *School Review*, 1903, pp. 727–41.

[17] "Year's Progress in the Mathematical Work of the University High School," G. W. Myers, *School Review*, 1907, pp. 576–93.

[18] "First-year Mathematics for Secondary Schools," G. W. Myers and others, University of Chicago Press.

In second-year mathematics geometry holds the center of attention, and arithmetical and algebraic elements are subordinated to it. The distinctive feature of the plan of presenting deductive geometrical truths consists of five general steps. The figure required by the demonstration is first sketched in the rough, in a way to exhibit clearly the conditions under which the truth in question is to be established. A careful drawing is next made on paper or on the blackboard with ruler and compass, under the specified conditions, and the appropriate parts of the figure that are drawn (protractor admitted) are carefully measured. Pupils are then required to make the best possible inferences as to the conclusions which follow from conforming to the imposed conditions. A correct enunciation of the principles to be established is next made and finally a deductive proof is given in standard form.

The mode of conducting the class work is a combination of the laboratory, the experimental, the Socratic and the class recitation modes. One of the advantages of the method is that it impresses the novice with the inadequacy of pure metrical means, and with the necessity of demonstrative methods.

As the result of six years' experience in teaching elementary high-school mathematics, Professor Lennes asserts his belief that graphical work is of great importance in creating interest and promoting a clearer and more satisfactory insight into subjects which too often are mysterious riddles.[19]

Professor Myers supplements this by saying that laboratory work with real problems, in the formulation and handling of which the pupil habituates himself to the transition from the concrete to the abstract, goes far toward supplementing the present isolated and abstract teaching of secondary mathematics.[20]

THE PHILOSOPHIC ATTITUDE

The reforms proposed by Professor Perry emphasized the practical features of instruction. In geometry especially there was a radical departure from Euclidean methods in the direction of the utilitarian. This tendency, however, is not universal. Objection is raised by a certain school of pure mathematicians to any system of mathematical instruction which is not severely logical, and which considers the subject as a means rather than an end. The following views of Professor Halsted may be considered as typical of this demand that mathematics be taught from the outset as a formal training in rigorous thinking.[21]

Halsted asserts that there must be a course in rational geometry which is really rigorous. This course should be founded on a preliminary course which does not strive to be necessarily demonstrative, but should emphasize the constructive phase. The purpose of the preliminary course should be, as Hailmann has said, to develop clear, geometrical notions, to give skill in accurate construction, to cultivate a healthy, esthetic feeling, and the power of visualizing creatively in geometrical design, thus stimulating genuine, vital interest in the study of geometry.

This preliminary course must fit the rigorous treatise on rational geometry which Halsted says should be written by some one familiar with the new, penetrating, critical researches in the principles of geometry.

Instead of agreeing with Professor Perry that many of the theorems in Euclid might well be assumed as axiomatic, Halsted as-

[19] "The Graph in High School Mathematics," N. J. Lennes, *School Review*, 1906.

[20] "Laboratory Equipment," Myers, *School Review*, 1903, pp. 727–41.

[21] Halsted, *Educational Review*, Vol. 24.

serts that greater rigor should be introduced, quoting Hilbert as saying that it is an error to believe that rigor in the proof is the enemy of simplicity.

With the new powers of insight given by the non-Euclidean geometry, and the introduction of Lobachevski's new principle in geometry, it was found that even Euclid made implicit assumptions. Thus, to make an angle congruent to a given angle involves a continuity assumption, while to prove other propositions requires a new set of assumptions which Halsted calls "betweenness assumptions," viz., of any three points of a straight line, there is always one, and only one, which lies between the other two.

The Euclidean method of superposition is also characterized as a worthless device; for if triangles are spatial but not material, there is a logical contradiction in the notion of moving them, while if they are material, they can not be perfectly rigid, and when superposed they are certain to be slightly deformed from the shape they had before.

Furthermore, so-called hypothetical constructions found in most text-books are criticized as illogical. Thus, certain propositions may require the construction of a regular heptagon or the trisection of an angle, although such constructions are impossible by elementary geometry. Thus, in many constructions, existential propositions are assumed. Helmholtz says of this: "In drawing any subsidiary line for the purpose of demonstration, the well-trained geometer asks always if it is possible to draw such a line."

This leads to the importance of not placing too great reliance upon diagrams. Bertram Russell says of Euclid I., that the first proposition assumes that the circles used in the construction intersect, an assumption not noticed by Euclid because of the dangerous habit of using figures. Hilbert believes in making frequent use of figures, but never depending upon them. The operations undertaken on a figure must always retain a purely logical validity. Halsted says that the beginners' course should consist largely in becoming familiar with figures, while in rational geometry that treatment of a proposition is best which connects it most closely with a visualization of the figures.

In rigorously founding a science, he believes that we should begin by setting up a system of assumptions containing an exact and complete description of the relations between the elementary concepts of this science. These axioms are at the same time the definitions of these elementary concepts. No statement within the science should be admitted as exact unless it can be derived from these assumptions by a definite number of logical deductions.

This criticism of the Perry movement and the laboratory method of instruction has recently been summarized by Professor Halsted as follows:[22]

> We knew that the so-called laboratory method for mathematics, the "measuring" method, was rotten at the core, since mathematics is not an experimental science, since no theorem of arithmetic, algebra or geometry can be proved by measurement; but, even granting the impossible, granting the super-human power of precise measurement, we could not thereby ever prove our space Euclidean, ever prove it the space taught in all our text-books.

THE PRACTICAL VIEW OF MATHEMATICS AS THE EXTENSION OF EXPERIENCE

One of the most sane and sensible views of the teaching of elementary mathematics yet presented is due to Professor Simon Newcomb.[23]

After recognizing the great difficulties

[22] "Even Perfect Measuring Impotent," Halsted, SCIENCE, October 25, 1907.

[23] Newcomb, *Educational Review*, Vol. 4.

inherent in the subject, Professor Newcomb goes on to say that in the teaching of elementary mathematics, especially arithmetic, care should be taken to embody mathematical ideas in a concrete form. The difficulty with the beginner is that he has no clear conception of the real significance of the subject which he is working upon. Figures and algebraic symbols do not represent to his mind anything which he can see or feel. So long as this continues his work consists of formal processes which have no correspondence in the world of sense.

Although a single experience may suffice to establish certain conceptions, it does not follow that the mind can apply these concepts in reasoning. It is one thing to know what a thing is but quite a different matter to handle it. This suggests that the difficulty in the teaching of elementary mathematics may be somewhat obviated by showing the mathematical relations among sensible objects.

To illustrate, not much progress was made in the study of imaginaries in algebra until Gauss and Cauchy conceived the idea of representing the two elements which enter into an algebraic imaginary by the position of a point in a plane. The motion of the point embodied the idea of the variation of the quantity, and the study of the subject was reduced to the study of the motion of points; in other words, an abstraction was replaced by a concrete representation. The result of this simple representation was that an extensive branch of mathematics was created, which would have been impossible if the abstract variable of algebra had not been replaced by the moving point of geometry. If such concrete representation is essential for expert mathematicians, it is obvious that immature pupils should be offered the same advantage.

In arithmetic, it is suggested that graphic methods be used throughout by way of illustration and explanation, lines being drawn to represent the numerical magnitude of the quantities involved. *Actual measurements, however, should not be made, but magnitudes should be estimated by the eye.*

This statement is especially noteworthy, as the idea implied seems to reconcile the differences between the ultra-practical and the ultra-logical extremes. The great danger in the laboratory method is that it will develop manual dexterity at the expense of intellectual power, or, from the ethical standpoint, that it will sacrifice the ideal to the material. The one pedagogical principle universally recognized, and the one on which it is claimed that the laboratory method is based, is that instruction should proceed from the concrete to the abstract. However, with the extensive laboratory equipment suggested by the advocates of the laboratory method, there must be a tendency, through lack of time if for no other reason, to remain with the concrete without making any sensible advance toward the abstract. Professor Newcomb's suggestion regarding the graphical depiction of relations without measurement, visualizes the idea to be presented without waste of time or involving the question of accuracy of measurement. The latter obviates the essential objection to graphical methods raised by Professor Halsted and others, and thus goes far toward meeting all demands, both critical and practical. Its simplicity, and the fact that this method has been, and is, in constant use, is also in its favor, and must appeal to all teachers who are interested in the progress of their pupils rather than in the exploitation of novel ideas.

It is also suggested that for at least one half the sums given in arithmetic, there be substituted a course of calculation of sizes,

weights and values of familiar things, such as, finding the dimensions of the schoolroom, the number of square feet in the floor and walls, the number of cubic feet in the room, the weight of the air in the room, the weight of the walls of the whole building, the number of bricks, etc. These would be more interesting than the complicated problems given in some of the advanced arithmetics.

Professor Newcomb deplores the fact that students who have taken a college course in physics can not compute the quantity of water which would be evaporated by the heat generated by the combustion of a ton of coal; or the number of cubic feet of air which could be warmed in the same way. He says that there is no good reason why this kind of elementary physics should not form a part of arithmetic, except adherence to traditional customs characteristic of the district school, and the prejudices of the so-called practical men against everything scientific in education.

In the study of geometry, the pupil should begin with constructive problems solved graphically; in beginning algebra, the pupil should first thoroughly familiarize himself with the use of symbolic language. Algebra is a kind of language, and to be proficient in its use this language must be learned by practise like any other unusual or foreign language.

Newcomb concludes by saying that it may be true that by adopting these suggestions the pupil would not get through any one book more rapidly and would make no better show of his knowledge upon examination. The advantages to be gained would be fewer courses, through fewer details of arithmetical applications being necessary, and a greater facility in the applications of arithmetic, algebra and geometry to practical questions.

A somewhat similar plea for practical mathematics is made by Fitzga in his work on natural methods of instruction.[24] The author first emphasizes the fact that in practise the use of mathematics arises from some external cause and that only concrete comprehensible things create a demand for its use, such as the coins, measures and weights in common use. The fact that numbers can not be seen, and that they are only phases of observation (*Beobachtungsmomente*) makes it necessary to present to the child's mind such objects, from which he can through observation fix numbers. In this selection much is gained if objects are chosen which will awaken interest. The things that interest the child most are those that are used in life, and such things as the child sees handled by adults. In the beginning of arithmetical instruction, therefore, numbers should be derived from different parts of the body, such as the fingers, eyes, etc., and later from arithmetical magnitudes (*Rechnungsgrössen*). These things are at first to be used for the purposes of observation lessons, for in this manner number presentations will be grasped without difficulty.

There is no question but that an exact knowledge of the mathematical magnitudes of life is necessary for the child, but the present methods of presentation do not permit of the exact observation of them. The child can not, through such a process, grasp the idea of magnitude, nor the relation of measures and their parts, and without this knowledge there can be no understanding of the subject. The observation of arithmetical magnitudes does not depend upon the physical properties of the objects observed, but upon the relation which they bear to one another. There are in each observation lesson certain vital features to

[24] "Die naturliche Methode des Rechenunterrichtes der Volks und Burgerschule," E. Fitzga, Wien, 1898.

which the observation of the child should be directed, and with such practise, numeration and figuring will follow of necessity. The aim of arithmetic should be chiefly to emphasize those arithmetical magnitudes which relate to practical life, and these should be presented to the child as objects of observation.

In order to determine the subject matter, it is necessary to consider that positive phases of the subject considered relate to practical life and in how far through the consideration of the subject matter can the moral and the cultural side of the child be nurtured.

The present lesson plans expect too much of the observational and comprehensive powers of the child, and it is certainly not a loss to the child if in the first-year arithmetic is not taught. In emphasizing the fundamental idea that numeration and arithmetical operations are chiefly to be acquired through the observations of arithmetical magnitudes, Fitzga expressly states that he does not approve of the rigorous handling of the subject matter. This, he says, naturally *assists the child in acquiring thoughtless habits.*

The conception of a thing only becomes clear when it is the abstraction of many similar and clear presentations. For this reason a teacher should not develop principles during a lesson in arithmetic, but should discourage the use of principles even by the brighter pupils, who have grasped these principles during the process of development, for the weaker pupils will immediately use them, and will not trouble themselves to acquire them through a process of logical reasoning. The teacher who wishes to bring his pupils to a clear comprehension of arithmetical problems can not develop any principles, but must wait until the child shall see the principles himself in the progress of his work.

Instead of the rigorous treatment of the subject, and the development of principles, Fitzga has, therefore, divided the separate parts of the subject matter into elements from which follows a logical arrangement based on the idea of repeated observations and presentations necessary to form clear conceptions of the different arithmetical operations.[25]

He also points out that it is necessary to choose examples in such a way that the context of the problem is comprehensible to the child. For this purpose it is necessary to bring to the mind of the child the various relations of practical life, out of which the examples are taken, and this can be done by giving a system of logical observation lessons.

The method of giving observational lessons as the basis of mathematical instruction has been explained in considerable detail by Jackman.[26] The article referred to is primarily a plea for the correlation of mathematics and physics, on the ground that mathematics has become so isolated that it is universally considered as the bugbear of the curriculum. This, it is stated, is due to the fact that the problems deal with subject matter with which the pupil has no concern, and mathematics has thus become simply a science of empty processes.

The question of the hour in the teaching of mathematics is not how do pupils in their thinking develop ideas of exact quantitative results, but ideas of what quantitative results are worth developing.

The unfortunate position in which mathematics finds itself is also partly due to the mistaken idea that a constant manipulation of mathematical formulas has a peculiar disciplinary value, whether they give any decidedly useful results or not. Great

[25] For courses in detail see his book, mentioned above.

[26] Jackman, *Educational Review,* Vol. 25.

mathematicians, however, have always acquired their disciplined powers while in the pursuit of knowledge having intrinsic worth, which indicates that mathematics should be related to actually useful and related things.

The past decade has seen a revolution in the schools. The old-time school with its barrenness of resource has been abolished, and the pupil has been placed in direct contact with all the vital activities of his time. To-day the child thinks through his hands, and it is currently believed that mathematics can play only a limited part in the new education. Yet since the whole universe is a manifestation of energy, mathematics must find its place in every subject. As a matter of fact, it is closely identified with physics and has already found its place in biology, botany, zoology, etc.

A radical change in the usual methods of presenting the mathematical branches must be made. Instead of taking them tandem fashion, the subjects of arithmetic, geometry and algebra must go hand in hand. The child solves the question for himself by introducing them all at once even before he enters school. It becomes then simply a question of assisting the pupil in the further development of the mathematical powers which he began to employ spontaneously before he came to school at all.

To illustrate the preceding principles and methods an outline of some work done in the eighth grade of the University Elementary School (Chicago) is given. The subject was botany and the pupils were allowed to take their time to work out the problems, as their observations demanded. In doing the work, the following principles were observed:

1. There must be a clear, general notion of the image to be developed.

2. There must be a careful selection of appropriate units of measurement.

3. The most expeditious methods of measurement or of applying the units must be chosen. Estimate first; then measure.

4. There must be a careful selection of processes by which the comparisons are made.

5. This must be followed by an objective representation of the results in the form of data obtained by observation. Gallons, quarts, pints, feet, yards, square feet, square yards, acres, miles, etc., must be seen until they become a part of the mental equipment.

6. Using the results obtained as data, a great nature picture must be constructed. That is to say, through the original and primary conception under which the pupil has been working, the real magnitude of world operations should be made to appear in definite quantitative results.

To illustrate these principles, the dispersal of seed was chosen as subject matter. The observation material in this case was found in a vacant city lot adjoining the school, and by extending the calculations to allied subjects, such as the amount of solar radiation, and of annual rainfall, the fundamental operations of arithmetic were thoroughly covered. The details of the work are fully explained in the last-mentioned article.

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WILLIAM EIMBECK, 1841–1909[1]

Mr. William Eimbeck, the subject of this sketch, and myself were close friends for many years. His ambitions were well known to me, and I am very well aware that his failure to attain the final success he had hoped for was due to an organic disease which slowly crept upon him during the later years of his life.

[1] Memorial address before the Philosophical Society of Washington, May 22, 1909.